Indeterminate secondary Hopf bifurcations in nonlinear oscillators

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A forced Van der Pol–type oscillator which exhibits secondary Hopf bifurcations is examined. Subcritical secondary Hopf bifurcations can arise when an unstable torus coalesces with a stable periodic cycle giving a jump to a disconnected attractor. We show that under the variation of a control parameter in the direction of instability *indeterminate subcritical secondary Hopf bifurcations* may occur where the outcome is extremely sensitive to any finite perturbation given to the system. [S1063-651X(97)08108-7]

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Self-excited oscillations occurs in many branches in physics and engineering [1]. For example, electronic circuits with nonlinear resistive properties often exhibit self-sustained oscillations [2]; aeroelastic flutter at high supersonic speeds can induce large amplitude oscillations [3]; dynamic instabilities of pipes conveying fluid [4–6]; buildings subjected to wind-induced oscillations [7]; and limit cycles produced by biochemical and chemical reactions [8].

In this paper we consider the determinacy of the outcome of a nonlinear oscillator that exhibits secondary Hopf bifurcations. We show that under the variation of a control parameter in the direction of instability, subcritical bifurcations may be indeterminate in which we cannot tell to which attractor the system may settle. Identifying such indeterminate subcritical secondary Hopf or Niemark bifurcations clearly contributes to our understanding of bifurcational phenomena. As described in a recent classification study [9], such a bifurcation was generically possible, but no example had yet been presented in the literature. Related studies on indeterminate Hopf bifurcations can be found in [10].

For systems modeled by nonlinear oscillators $\ddot{x}+\mu\dot{x}+\omega^2x+f(x,\dot{x})=0$, where $f(x,\dot{x})$ contains the nonlinear terms, one mechanism by which the equilibrium state of the system (at $x=\dot{x}=0$) can lose its stability is the *Hopf bifurcation*; in this case it is associated with the vanishing of linear damping in the motion of an oscillator. If $f(x,\dot{x})=0$ the system is linear and the postcritical motion is typified by oscillations whose amplitude grows to infinity. The addition of nonlinear terms can destroy this feature. Depending upon the stabilizing or destabilizing nature of $f(x,\dot{x})$ in the vicinity of the equilibrium state, *supercritical* Hopf or *subcritical* Hopf bifurcations may occur. If one were to inspect the mapping eigenvalues at the point of instability there would be a pair of complex conjugate eigenvalues which cross the unit circle.

Supercritical Hopf bifurcations are continuous and always determinate, in which the loss of stability of the equilibrium state gives rise to a limit cycle; in effect this form of bifur-

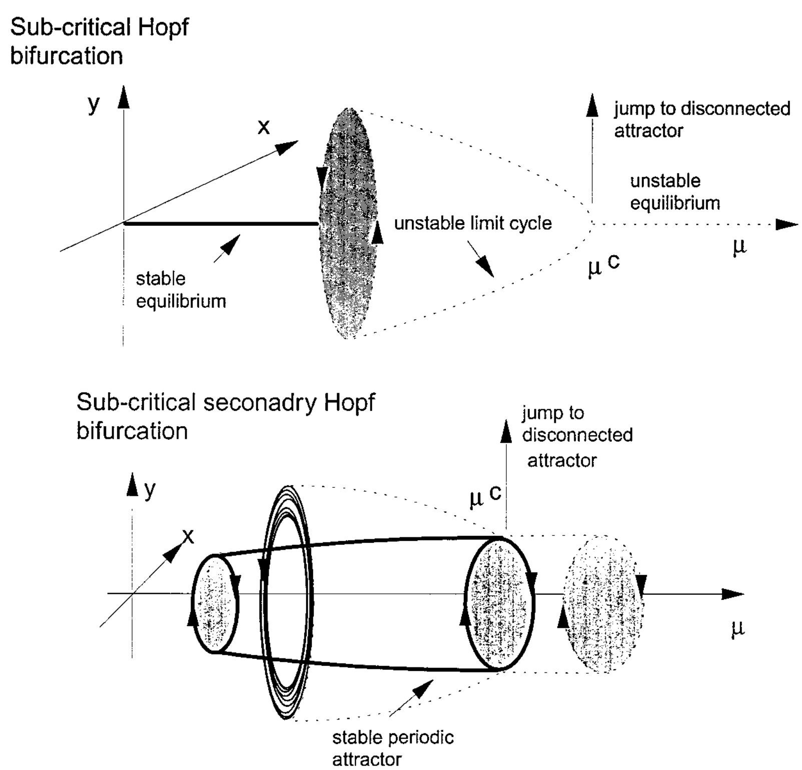

FIG. 1. Schematic representation of secondary Hopf or Niemark bifurcation.

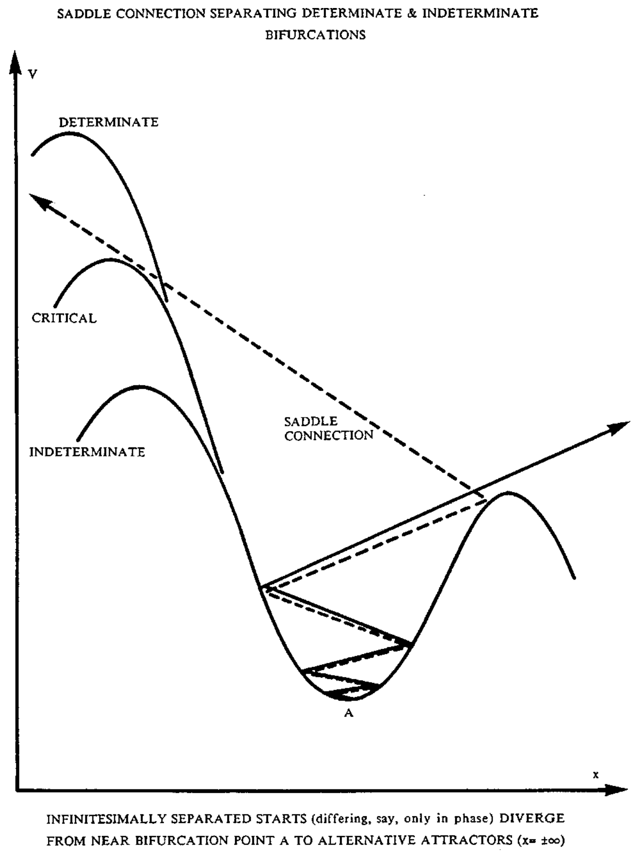

FIG. 2. Schematic representation of the saddle of connection.

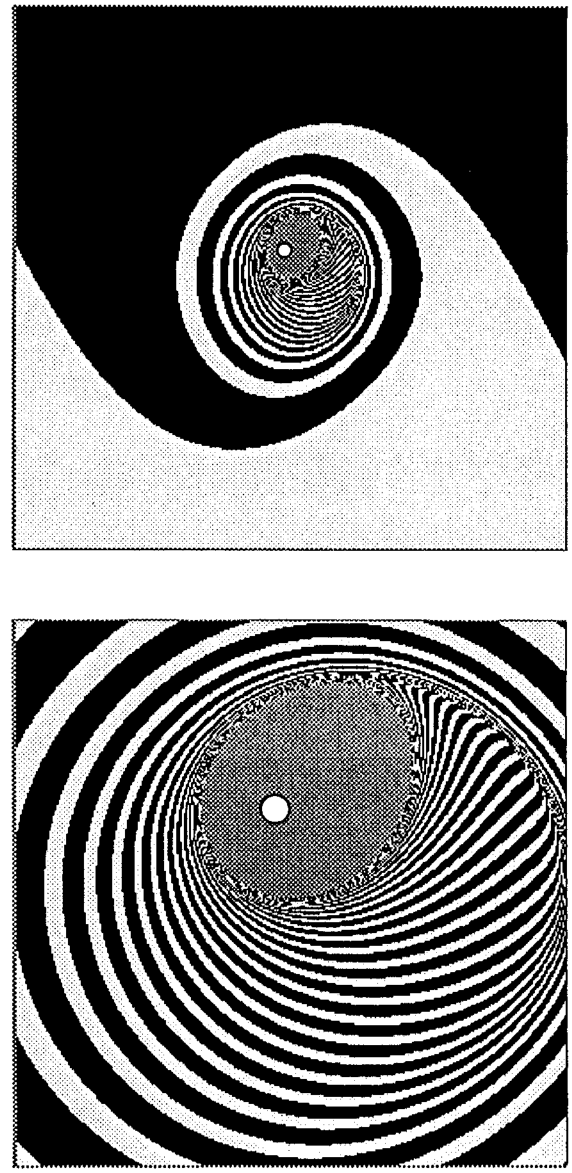

FIG. 3. Indeterminate subcritical bifurcations. (a) Attractor-basin phase portrait just prior to an indeterminate subcritical bifurcation in the window $-1.2<x<1.2$, $-1.0<y<1.0$. Here $F=0.01$, $\mu=0.03$, $D=0.9$. Gray shading indicates the basin of attraction of the period-one solution; white shading for the basin of attraction of $+$infinity, black of $-$infinity. The dot represents the attractor. (b) A blow up of (a) in the window $-0.3<x<0.3$, $-0.3<y<0.3$.

cation of an equilibrium point can be categorized as the separation point between an area contracting region with positive damping, and an area expanding region due to negative damping, at which the previously stable equilibrium point becomes unstable, and a limit cycle develops. On the other hand, subcritical Hopf bifurcations, as shown in Fig. 1(a), are discontinuous. The equilibrium which is stable for $\mu<\mu_c$ becomes unstable at $\mu=\mu_c$; this occurs where there is coalescence of an unstable limit cycle with the stable equilibrium state. Here any small perturbation from the equilibrium state results in a jump to a disconnected attractor.

We may extend the bifurcations of equilibria to bifurcations of cycles by considering a three dimensional differential equation system $\dot{x}=F(x,\mu)$, exhibiting periodic cycles $\mu<\mu_c$. Suppose that at these parameter values the system is locally dissapative. If μ were to increase to μ_c, where the system is locally conservative, while considering that for $\mu<\mu_c$ it is area expanding in the neighborhood of the periodic cycle, at $\mu=\mu_c$ a bifurcation analogous to the supercritical Hopf bifurcation takes place [11]. This bifurcation is often called a *secondary Hopf bifurcation* or a *Niemark bifurcation*. Subcritical forms of this bifurcation occur when destabilizing nonlinearities dominate as the periodic cycle loses its stability. Figure 1(b) shows a subcritical secondary Hopf bifurcation. Here an unstable torus coalesces with a stable periodic cycle at $\mu=\mu_c$ leaving an unstable periodic cycle for $\mu>\mu_c$.

We shall consider the example of a sinusiodally forced nonlinear oscillator [12]

$$\ddot{x}+\mu\dot{x}+\omega^2x+f(x,\dot{x})=F\ \sin\omega t.$$

Various authors have studied this type of problem: Cartwright and Littlewood [13] showed that competing subharmonic responses may occur at the same set of parameters; Hayashi [14] showed that for small forcing levels the oscillator does not lock onto the forcing frequency. Here the steady state is compounded with two frequencies, the natural frequency and the forcing frequency exhibiting a quasiperiodic response; Guckenheimer and Holmes [15] outlined bifurcation diagrams showing the possibility of chaotic behavior; Shaw [16] observed steady state chaotic behavior.

In particular, we consider a periodically forced oscillator with nonlinear damping and nonlinear stiffness characteristics that has the ability to escape from a asymmetric single potential well, $V(x)=x^2/2+(D-1)x^3/3-Dx^4/4$

$$\ddot{x}+\mu\dot{x}-\dot{x}^3+x+(D-1)x^2-Dx^3=F\ \sin\omega t, \quad \dot{x}\equiv y.$$

We shall consider the case where $F=0.01$ and $\omega=0.95$. For relatively large values of μ there is a stable periodic attractor S_n that orginates from $F=x=y=0$. As we reduce the control μ this periodic oscillation loses its stability at a subcritical secondary Hopf bifurcation beyond which the divergence is positive and all trajectories diverge to $x=\pm\infty$. As regards the basin structure, the basin shrinks around the attractor and pinches it off at the critical control parameter; the size of the basin drops continuously to zero as the bifurcation is approached. The nonlinear stiffness corresponds to a metastable well similar to that in Fig. 2. The parameter D allows us to vary the relative heights of the two potential barriers. Taking the left hand barrier to be the higher, as drawn, it is clear that if the heights are very different then the bifurcation will be determinate, with all motions tending to $x=\pm\infty$. Conversely, if the heights are nearly the same, the bifurcation will be indeterminate, with adjacent motions differing perhaps in their starting phase heading to $x=\pm\infty$. Figure 3(a) shows the basin structures in the Poincaré section just before the bifurcation with Fig. 3(c) showing a blow up in the vicinity of the attractor. In our example we have chosen $\mu=0.03$ with $\mu_c\approx0.016$. The critical condition, given on separating these two forms of bifurcations, is represented by the *saddle connection*, in which a trajectory climbing out of the well under positive divergence can just pass from the lower hilltop to the higher hilltop. This is termed a *regular indeterminate bifurcation* (the generation of such a bifurcation is described in [15,17,18]). Here basins accumulate, with vanishing thickness, onto the unstable limit cycle, in what is termed by Stewart and Ueda [18] a ''mosquito coil structure.'' The outcome of this local bifurcation is clearly unpredictable; in any real situation, due to the inherent uncertainties in the specification of the initial conditions or parameter values, long term predictability will be lost and hence the jump will be indeterminate.

In summary, we have described the phenomenon of indeterminate subcritical secondary Hopf or Niemark bifurcations. We have shown that when such bifurcations occur we cannot tell to which attractor the system may settle. Identifying such an indeterminate subcritical secondary Hopf bifurcations clearly contributes to our understanding of bifurcational phenomena since no example has yet been presented in the literature.

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